

2.2 – Evaluating Determinants by Row Reduction

Theorem 2.2.1 Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

Theorem 2.2.2 Let A be a square matrix. Then $\det(A) = \det(A^T)$.

Theorem 2.2.3 Elementary Row Operations

Let A be an $n \times n$ matrix.

- a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k , then $\det(B) = k \det(A)$.
- b) If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.
- c) If B is the matrix that results when a multiple of one row of A is added to another or when a multiple of one column is added to another, then $\det(B) = \det(A)$.

pf (a): Let B be the matrix that results from multiplying the i^{th} row of A by k .
 $\det(A) = \sum_{l=1}^n a_{il} C_{il}$ by expanding along the i^{th} row
 $\det(B) = \sum_{l=1}^n b_{il} C_{il} = \sum_{l=1}^n k a_{il} C_{il} = k \sum_{l=1}^n a_{il} C_{il} = k \det(A)$

Consider $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 6 & 8 \\ 5 & 7 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \\ 5 & 7 & 2 \end{vmatrix}$

Note: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$ but $2 \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 2(ad-bc)$

Theorem 2.2.4 Let E be an $n \times n$ elementary matrix.

- a) If E results from multiplying a single row of I_n by a nonzero number k , then $\det(E) = k$.
- b) If E results from interchanging two rows of I_n , then $\det(E) = -1$.
- c) If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.

This is a special case of Thm 2.2.3

#12 Evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion.

$$\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + \underline{\hspace{2cm}}$$

$$R_3 \rightarrow \cancel{2R_3} + \underline{\hspace{2cm}}$$

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \underline{17/2} \end{vmatrix}$$

$$R_2 \rightarrow R_2 + 2R_1$$

$$R_3 \rightarrow R_3 + \frac{13}{2}R_2$$

$$= -17$$

$$R_3 \rightarrow R_3 - 5R_1$$

$$= (-2)\left(\frac{17}{2}\right) \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{vmatrix} = \underline{-17}$$

In #16 and 20, evaluate the determinant, given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$.

$$\text{\#16} \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} \quad R_1 \leftrightarrow R_3$$

$$\text{\#20} \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} \quad \begin{matrix} R_2 \rightarrow 2R_2 \\ R_3 \rightarrow R_3 + 3R_1 \end{matrix}$$

$$= -(-6) = 6$$

$$= -12$$

Theorem 2.2.5 If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

A row operation would yield a row of zeros. $\rightarrow$ see Thm 2.2.1.