Recall: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

2.1 – Determinants by Cofactor Expansion

Definition: If A is a square matrix, then the **minor of entry a_{ij}** is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the **cofactor of entry a_{ij}**.

$A = [a_{ij}]$ $\quad C_{ij} = (-1)^{i+j} M_{ij}$

$1+1=2$ $\quad 1+2=3$

Example: Find all the minors and cofactors of $A = \begin{bmatrix} \overset{+}{2} & \overset{-}{3} & \overset{+}{-7} \\ \overset{-}{4} & \overset{+}{-2} & \overset{-}{-9} \\ \overset{+}{2} & \overset{-}{5} & \overset{+}{6} \end{bmatrix}$. ← Matrix notation

(Notation) determinant: $\begin{vmatrix} 2 & 3 & -7 \\ 4 & -2 & -9 \\ 2 & 5 & 6 \end{vmatrix}$

Minors

$M_{11} = \begin{vmatrix} -2 & -9 \\ 5 & 6 \end{vmatrix} = -12-(-45) = 33$ $\quad M_{12} = \begin{vmatrix} 4 & -9 \\ 2 & 6 \end{vmatrix} = 42$ $\quad M_{13} = \begin{vmatrix} 4 & -2 \\ 2 & 5 \end{vmatrix} = 24$

$C_{11} = 33$ $\quad C_{12} = -42$ $\quad C_{13} = 24$

$M_{21} = \begin{vmatrix} 3 & -7 \\ 5 & 6 \end{vmatrix} = 53$ $\quad M_{22} = \begin{vmatrix} 2 & -7 \\ 2 & 6 \end{vmatrix} = 26$ $\quad M_{23} = \begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} = 4$

$C_{21} = -53$ $\quad C_{22} = 26$ $\quad C_{23} = -4$

$M_{31} = \begin{vmatrix} 3 & -7 \\ -2 & -9 \end{vmatrix} = -41$ $\quad M_{32} = \begin{vmatrix} 2 & -7 \\ 4 & -9 \end{vmatrix} = 10$ $\quad M_{33} = \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = -16$

$C_{31} = -41$ $\quad C_{32} = -10$ $\quad C_{33} = -16$

Definition: If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the **determinant of A**, and the sums themselves are called **cofactor expansions of A**. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} \text{ (cofactor expansion along the } j\text{th column)}$$
$$= a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} \text{ (cofactor expansion along the } i\text{th row)}$$

Example: Compute the determinant of the matrix A above.

Matrix $A = \begin{bmatrix} 2 & 3 & -7 \\ 4 & -2 & -9 \\ 2 & 5 & 6 \end{bmatrix}$

Using the wrong symbol is incorrect

Determinant: $\det(A) = |A| = \begin{vmatrix} 2 & 3 & -7 \\ 4 & -2 & -9 \\ 2 & 5 & 6 \end{vmatrix}$

$= 2(33) - 3(42) \quad -7(24)$

$= -228$

#11 Use the arrow technique of Figure 2.1.1 to evaluate the determinant.

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} \begin{matrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{matrix} = 45 - 110 = -65$$

$20 + 84 + 6 = 110$

$-20 - 7 + 72 = 45$

Arrow method for 2×2 determinants:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Why?

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Theorem 2.1.1 If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

#16 Find all values of λ for which $\det(A) = 0$.

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

$$(\lambda - 4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda - 4)(\lambda^2 - \lambda - 6) = 0$$

$$(\lambda - 4)(\lambda - 3)(\lambda + 2) = 0$$

$$\lambda = -2, 3, 4$$

Theorem 2.1.2 If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

#31 Evaluate the determinant of the given matrix by inspection.

$$A = \begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{vmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & -4 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= 1\left(1\begin{vmatrix} 2 & 7 \\ 0 & 3 \end{vmatrix}\right)$$

$$= 1(1)(2)(3)$$

$$\det(A) = 1(1)(2)(3)$$
$$= 6$$

Aside: $A = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 & 8 & 7 \\ 5 & -3 & 2 \\ 8 & 12 & 17 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 4 & 8 & 7 \\ 5 & -3 & 2 \\ 8 & 12 & 17 \end{bmatrix}$ Not defined

$\det(A)\det(B) = \begin{vmatrix} 2 & 1 \\ -3 & 5 \end{vmatrix} \begin{vmatrix} 4 & 8 & 7 \\ 5 & -3 & 2 \\ 8 & 12 & 17 \end{vmatrix}$ is defined