1.8 – Introduction to Linear Transformations

xy-plane 3-space

Definitions: $\mathbf{R}^2$ is the set of all ordered pairs, $\mathbf{R}^3$ is the set of all ordered triples, and $\mathbf{R}^n$ is the set of all ordered n-tuples. Elements of R^n are called **vectors**.

The **standard basis vectors** for R^n are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, ..., $\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$. These can also be written as row vectors if convenient.

All vectors in R^n can be expressed as linear combinations of these vectors:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

Definitions: A **function** f is a rule that associates with an input (an element of the **domain** of f) a unique output (an element of the **codomain** of f). The output is the **image** of the input, and the set of all images is called the **range** of f. Note that the range of f is a subset of the codomain of f.

$f(a) = b$ ← the image of a under f

$a \in$ domain of f , $b \in$ codomain of f

Consider $f(x) = |x|$

Domain: R

Codomain: R

Range: $[0, \infty)$

Transformations

Consider the system $\begin{aligned} 4 &= 2x - 3y + 5z \\ 6 &= 7x + 2y - z \end{aligned}$

In matrix form this is

$$\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ 7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The 2×3 matrix $A = \begin{bmatrix} 2 & -3 & 5 \\ 7 & 2 & -1 \end{bmatrix}$ maps vectors in R^3 into vectors in R^2.

A is called a matrix transformation from R^3 to R^2.

Definition: If T is a function with domain R^n and codomain R^m, then we say that T is a **transformation** from R^n to R^m or that T **maps** from R^n to R^m, which we denote by writing $T : R^n \to R^m$. In the special case where $m = n$, a transformation is sometimes called an **operator** on R^n.

#5 Find the domain and codomain of the transformation defined by the matrix product.

a. $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$2 \times 3 \quad 3 \times 1 \to 2 \times 1$

b. $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

a) Domain: R^3

Codomain: R^2

b) Domain: R^2

Codomain: R^3

Definitions: A **matrix transformation** $\mathbf{w} = A\mathbf{x}$ maps a vector $\mathbf{x} \in R^n$ to a vector $\mathbf{w} \in R^m$ by multiplying $\mathbf{x}$ on the left by A [which is an $m \times n$ matrix]. If $m = n$, then we call the transformation a **matrix operator**. A matrix transformation is denoted $T_A : R^n \to R^m$ or $\mathbf{w} = T_A(\mathbf{x})$ if we do not need to specify the domain and codomain. This can also be written in the form

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w}$$

verbalized as "T_A maps $\mathbf{x}$ into $\mathbf{w}$." The matrix A is the **standard matrix** for the transformation.

#8 Find the domain and codomain of the transformation T defined by the formula.

a. $T(x_1, x_2, x_3, x_4) = (x_1, x_2)$ $T(3,4,5,6) = (3,4)$

b. $T(x_1, x_2, x_3) = (\underline{x_1}, \underline{x_2 - x_3}, \underline{x_2})$ $T(5,6,7) = (5,-1,6)$

a) Domain: R^4 Codomain: R^2

b) Domain: R^3 Codomain: R^3 (operator)

#12 Find the standard matrix for the transformation defined by the equations.

a.
$$\begin{aligned} w_1 &= -x_1 + x_2 \\ w_2 &= 3x_1 - 2x_2 \\ w_3 &= 5x_1 - 7x_2 \end{aligned}$$

b.
$$\begin{aligned} w_1 &= x_1 \\ w_2 &= x_1 + x_2 \\ w_3 &= x_1 + x_2 + x_3 \\ w_4 &= x_1 + x_2 + x_3 + x_4 \end{aligned}$$

$$\vec{w} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

#13 Find the standard matrix for the transformation T defined by the formula.
a. $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$ $\Rightarrow T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$
b. $T(x_1, x_2, x_3) = (4x_1 + x_2, x_1 + x_2)$

a) $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Definitions: The **zero transformation** from R^n to R^m, $T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$, maps every vector in R^n to the zero vector in R^m. The **identity operator** $T_{I_n}(\mathbf{x}) = I_n\mathbf{x} = \mathbf{x}$ maps every vector in R^n to itself.

$T_A(\vec{x}) = A\vec{x}$

Theorem 1.8.1 Properties of Matrix Transformations
For every matrix A the matrix transformation $T_A : R^n \to R^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:
a) $T_A(\mathbf{0}) = \mathbf{0}$
b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (homogeneity property)
c) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (additivity property) $\quad 3(x+y) = 3x+3y$
d) $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$ $\quad (x+y)^2 \neq x^2+y^2$

These follow from properties of matrix arithmetic

Theorem 1.8.2 $T : R^n \to R^m$ is a matrix transformation if and only if the following relationships hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:
i) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (additivity property)
ii) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (homogeneity property)

($\Rightarrow$) From properties of matrix arithmetic

Pf: ($\Leftarrow$) Assume (i) & (ii) hold. The proof will be complete if there is an $m \times n$ matrix

A such that $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in R^n$.

By (i) and (ii), $T(k_1\vec{u}_1 + k_2\vec{u}_2 + \cdots + k_n\vec{u}_n)$

$$= k_1T(\vec{u}_1) + k_2T(\vec{u}_2) + \cdots + k_nT(\vec{u}_n)$$

for all vectors $\vec{u}_i \in R^n$ and all scalars $k_i \in R$.

Let $A = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n)\right]$, where

$\vec{e}_i$ is a standard basis vector for R^n.

Then

$$A\vec{x} = x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n),$$

a linear combination of the column vectors of A in which coefficients are entries of $\vec{x}$.

We then have $A\vec{x} = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$

$$= T(\vec{x})$$

by linearity of T.

Definitions: The properties (i) and (ii) in the preceding theorem are called **linearity conditions**. A transformation that possesses these properties is a **linear transformation**.

Vocabulary:

Theorem 1.8.3 Every linear transformation from R^n to R^m is a matrix transformation, and conversely every matrix transformation from R^n to R^m is a linear transformation.

Theorem 1.8.4 If $T_A : R^n \to R^m$ and $T_B : R^n \to R^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A = B$.

(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

(ii) $T(k\vec{u}) = k\,T(\vec{u})$

#22 Use Theorem 1.8.2 to show that T is a matrix transformation.

a. $T(x,\ y,\ z) = (x + y,\ y + z,\ x)$ $\quad T: R^3 \to R^3$

b. $T(x_1, x_2, x_3) = (x_1, x_3, x_1 + x_2)$

a. Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, $k \in R$.

$\vec{u}+\vec{v} = (u_1+v_1,\ u_2+v_2,\ u_3+v_3)$

$T(\vec{u}+\vec{v}) = T(u_1+v_1,\ u_2+v_2,\ u_3+v_3)$

$= (u_1+v_1+u_2+v_2,\ u_2+v_2+u_3+v_3,\ u_1+v_1)$

$T(\vec{u})+T(\vec{v}) = (u_1+u_2,\ u_2+u_3,\ u_1) + (v_1+v_2,\ v_2+v_3,\ v_1)$

$= (u_1+u_2+v_1+v_2,\ u_2+u_3+v_2+v_3,\ u_1+v_1)$

So $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$

$T(k\vec{u}) = T(ku_1, ku_2, ku_3) = (ku_1+ku_2,\ ku_2+ku_3,\ ku_1)$

$= k(u_1+u_2,\ u_2+u_3,\ u_1)$

$= k\,T(\vec{u})$

T is a matrix transformation

23. Use Theorem 1.8.2 to show that T is not a matrix transformation.

a. $T(x,\ y) = (x^2, y)$

b. $T(x,\ y,\ z) = (x,\ y,\ xz)$

a. $T(kx, ky) = (k^2x^2, ky)$
$= k(kx^2, y)$
$\neq kT(x,y)$

OR let $k=5$, $x=3$, $y=4$.
Then $T(5\cdot3, 5\cdot4) = (25\cdot9, 5\cdot4) = (225, 20)$
$5\,T(3,4) = 5(9,4) = (45, 20) \neq (225, 20)$

The $(\Leftarrow)$ of the proof of thm 1.8.2 suggests a technique for finding the standard matrix for a transformation: Find what it does to standard basis vectors – these are the columns

#28 The images of the standard basis vectors for R^3 are given for a linear transformation $T : R^3 \to R^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$.

A

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix}$$

Recall

#5 Find the domain and codomain of the transformation defined by the matrix product.

a. $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 + 2x_3 \\ 6x_1 + 7x_2 + x_3 \end{bmatrix}$$

$$T_A(\vec{x}) = (3x_1 + x_2 + 2x_3,\ 6x_1 + 7x_2 + x_3)$$

$$A\vec{e}_1 = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{\begin{bmatrix} 3 \\ 6 \end{bmatrix}} = T_A(\vec{e}_1)$$

$$A\vec{e}_2 = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underline{\begin{bmatrix} 1 \\ 7 \end{bmatrix}} = T_A(\vec{e}_2)$$

$$A\vec{e}_3 = \underline{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = T_A(\vec{e}_3)$$

$$A = \left[T_A(\vec{e}_1) \mid T_A(\vec{e}_2) \mid T_A(\vec{e}_3) \right]$$

Example: Find the standard matrix A for the linear transformation $T : R^2 \to R^3$ for which

$$T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\1\\0\end{bmatrix} \text{ and } T\left(\begin{bmatrix}3\\-5\end{bmatrix}\right) = \begin{bmatrix}5\\-7\\1\end{bmatrix}$$

and use it to compute $T\left(\begin{bmatrix}-4\\3\end{bmatrix}\right)$.

Thought process:

Let $\vec{u} = \begin{bmatrix}-1\\2\end{bmatrix}$, $\vec{v} = \begin{bmatrix}3\\-5\end{bmatrix}$.

If we can find a, b such that $\vec{e}_1 = a\vec{u} + b\vec{v}$, then we can form $T(\vec{e}_1) = aT(\vec{u}) + bT(\vec{v})$. Likewise for $\vec{e}_2$.

We then want $a, b \ni a\vec{u} + b\vec{v} = \vec{e}_1 \overset{\text{OR}}{=} \vec{e}_2$

$$\Rightarrow a\begin{bmatrix}-1\\2\end{bmatrix} + b\begin{bmatrix}3\\-5\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \overset{\text{OR}}{=} \begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-a+3b\\2a-5b\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \overset{\text{OR}}{=} \begin{bmatrix}0\\1\end{bmatrix} \rightarrow \left[\begin{array}{cc|cc}-1 & 3 & 1 & 0\\2 & -5 & 0 & 1\end{array}\right]$$

Solving both systems simultaneously

$$\Rightarrow \left[\begin{array}{cc|cc}1 & 0 & 5 & 3\\0 & 1 & 2 & 1\end{array}\right]$$

a & b that give $\vec{e}_1$ | a & b that give $\vec{e}_2$

$\vec{e}_1 = 5\vec{u} + 2\vec{v}$ | $\vec{e}_2 = 3\vec{u} + \vec{v}$

$\Rightarrow T(\vec{e}_1) = 5T(\vec{u}) + 2T(\vec{v})$ | $T(\vec{e}_2) = 3T(\vec{u}) + T(\vec{v})$

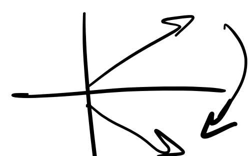
$T(\vec{e}_1) = 5\begin{bmatrix}2\\1\\0\end{bmatrix} + 2\begin{bmatrix}5\\-7\\1\end{bmatrix}$ | $T(\vec{e}_2) = 3\begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}5\\-7\\1\end{bmatrix}$

$T(\vec{e}_1) = \begin{bmatrix}20\\-9\\2\end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix}11\\-4\\1\end{bmatrix}$

$$A = \begin{bmatrix} 20 & 11 \\ -9 & -4 \\ 2 & 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 20 & 11 \\ -9 & -4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -47 \\ 24 \\ -5 \end{bmatrix}$$

Matrix operators on R^2

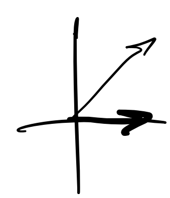


Reflection operators

- Reflection about the x-axis: $T(x,\ y) = (x,\ -y)$, $T_A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Reflection about the y-axis: $T(x,\ y) = (-x,\ y)$, $T_A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- Reflection about the line $y = x$: $T(x,\ y) = (y,\ x)$, $T_A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Projection operators

- Orthogonal projection onto the x-axis: $T(x,\ y) = (x,\ 0)$,
  $T_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

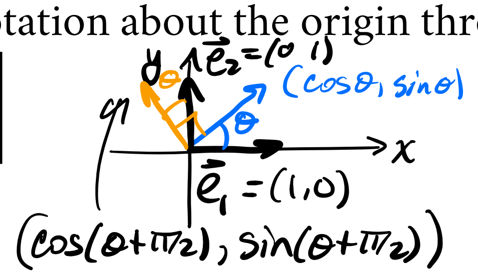


- Orthogonal projection onto the y-axis: $T(x,\ y) = (0,\ y)$,
  $T_A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

Rotation operator

- Counterclockwise rotation about the origin through an angle θ:
  $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$T(\vec{e}_1)$ $T(\vec{e}_2)$



$\cos(\theta+\pi/2)$

$= \cos\theta \cos\frac{\pi}{2} - \sin\theta \sin\frac{\pi}{2}$ (0, 1)

$= -\sin\theta$

Matrix operators on R^3

Reflection operators

- Reflection about the xy-plane: $T(x,\ y,\ z) = (x,\ y,\ -z)$,
$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
- Reflection about the xz-plane: $T(x,\ y,\ z) = (x,\ -y,\ z)$,
$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- Reflection about the yz-plane: $T(x,\ y,\ z) = (-x,\ y,\ z)$,
$$T_A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Projection operators

- Orthogonal projection onto the xy-plane: $T(x,\ y,\ z) = (x,\ y,\ 0)$,
$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
- Orthogonal projection onto the xz-plane: $T(x,\ y,\ z) = (x,\ 0,\ z)$,
$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- Orthogonal projection onto the yz-plane: $T(x,\ y,\ z) = (0,\ y,\ z)$,
$$T_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$