1.5 – Elementary Matrices and a Method for Finding A^{-1}

Definition: Matrices A and B are said to be **row equivalent** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

useful for developing theory

Definition: A matrix E is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

Theorem 1.5.1 Row Operations by Matrix Multiplication
If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

#6 An elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A.

a. $E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$R_1 \to -6R_1$

b. $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

c. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

a) $EA = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix} = \underline{\begin{bmatrix} 6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{bmatrix}}$

$A : R_1 \to -6R_1 \quad \underline{\begin{bmatrix} 6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{bmatrix}}$

b) $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Row operation: $R_2 \to R_2 - 4R_1$

__pf__ of Thm 1.5.1, Case I: E results from multiplying the i^{th} row by $k \neq 0$.

Let $\vec{e}_1 = [1 \ 0 \cdots 0]$, $\vec{e}_2 = [0 \ 1 \cdots 0]$, $\vec{e}_m = [0 \ 0 \cdots 1]$, $k\vec{e}_i = [0 \ 0 \cdots k \cdots 0]$

$$EA = \begin{bmatrix} \vec{e}_1 A \\ \vec{e}_2 A \\ \vdots \\ k\vec{e}_i A \\ \vdots \\ \vec{e}_m A \end{bmatrix}$$, which results in multiplying the i^{th} row of A by k.

#7 Use the following matrices and find an elementary matrix E that satisfies the stated equation.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix},$$

$$D = \begin{bmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{bmatrix}, F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

c. $EA = C$

To get from A to C, $R_3 \to R_3 - 2R_1$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 1.5.2 Every elementary matrix is invertible, and the inverse is also an elementary matrix.

pf: Let E be the elementary matrix that results from performing a single row operation on I and let E' be the matrix that results from performing the inverse operation on I.

Then by Thm 1.5.1, $EE' = I$

and $E'E = I$ so $E' = E^{-1}$.

Dependency chain: Def I_n, Def row ops $\Rightarrow$ Def elem mat. $\rightarrow$ Thm 1.5.1 $\rightarrow$ Thm 1.5.2

Def of A^{-1} $\rightarrow$ Thm 1.5.2

Theorem 1.5.3 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all are true or all false.

~~$a \Leftrightarrow b \Leftrightarrow c \Leftrightarrow d$~~

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

$$\begin{tikzcd} & a \arrow[dr] & \\ d \arrow[ur] & & b \arrow[dl] \\ & c \arrow[ul] & \end{tikzcd}$$

there exists / such that

pf: $a \Rightarrow b$ A invertible $\Rightarrow \exists A^{-1} \ni AA^{-1} = I$.

$$A\vec{x} = \vec{0} \Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{0} \Rightarrow \vec{x} = \vec{0}$$

$$I\vec{x} = A^{-1}\vec{0} \Rightarrow \vec{x} = A^{-1}\vec{0}$$

by Thm 1.4.2 d.

If $\vec{x}'$ is any other solution, then

$A\vec{x} = \vec{0}$ and $A\vec{x}' = \vec{0}$

$$\Rightarrow A\vec{x} - A\vec{x}' = \vec{0} \Rightarrow A(\vec{x} - \vec{x}') = \vec{0}$$

$$\vec{x} - \vec{x}' = \vec{0} \Rightarrow \vec{x} = \vec{x}'.$$

$A\vec{x} = \vec{0}$ has <u>only</u> the trivial solution.

$b \Rightarrow c$ $A\vec{x} = \vec{0}$ has only the trivial solution, so the system contains no free variables.

By Thm 1.2.1, rref has n nonzero rows.

Then by Thm 1.4.3 rref is I_n.

$c \Rightarrow d$ rref of A is I_n, so I_n can be obtained from A by a series of row operations. We can express $I_n = E_r \cdots E_2 E_1 A$ by Thm 1.5.1.

Then $(E_r \cdots E_2 E_1)^{-1} (E_r \cdots E_2 E_1) A = (E_r \cdots E_2 E_1)^{-1} I_n$

$A = E_1^{-1} E_2^{-1} \cdots E_r^{-1}$, which are elementary matrices by Thm 1.5.2.

$d \Rightarrow a$ Since $I_n = E_r \cdots E_2 E_1 A$ from the previous step, $E_r \cdots E_2 E_1 = A^{-1}$.

Thus A is invertible.

Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

#15 $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

The inversion algorithm is based on the $c \Rightarrow d$ proof above.

Since $I_n = E_r \cdots E_2 E_1 A$, we have $A^{-1} = E_r \cdots E_2 E_1 I_n$. So the elementary row operations that reduce A to I_n

will yield A^{-1} when applied to I_n.

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \Rightarrow R_2 - R_1$

$$\begin{array}{cccccc} 2 & 7 & 6 & 0 & 1 & 0 \\ -2 & -6 & -6 & -1 & 0 & 0 \\ \hline 0 & 1 & 0 & -1 & 1 & 0 \end{array}$$

$R_3 \Rightarrow R_3 - R_1$

$$\begin{array}{cccccc} 2 & 7 & 7 & 0 & 0 & 1 \\ -2 & -6 & -6 & -1 & 0 & 0 \\ \hline 0 & 1 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \Rightarrow R_3 - R_2$

$$\begin{array}{cccccc} 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$R_1 \Rightarrow R_1 - 6R_3$

$$\begin{array}{cccccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 & 6 & -6 \\ \hline 2 & 6 & 0 & 1 & 6 & -6 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$R_1 \Rightarrow R_1 - 6R_2$

$$\begin{array}{cccccc} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & -6 & 0 & 6 & -6 & 0 \\ \hline 2 & 0 & 0 & 7 & 0 & -6 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_1 \Rightarrow \tfrac{1}{2}R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

In general, $[A \mid I_n] \xrightarrow[\text{ops}]{\text{Elem. row}} [I_n \mid A^{-1}]$

#18 $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$ Good practice

#20 Find the inverse of each of the following 4×4 matrices, where k_1, k_2, k_3, k_4, k are all nonzero.

a. $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_4$
$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{cccc|cccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right]$$

Multiply: $\frac{1}{k_i} R_i$

$$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/k_4 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 1/k_1 & 0 & 0 & 0 \end{bmatrix}$$

#27 Show that the matrices A and B are row equivalent by finding a sequence of elementary row operations that produces B from A, and then use that result to find a matrix C such that $CA = B$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduce A to B, keeping track of E_i. corresponding elementary matrix

$R_2 \to R_2 - R_1$

$$\begin{array}{rrr} 1 & 4 & 1 \\ -1 & -2 & -3 \\ \hline 0 & 2 & -2 \end{array} \checkmark$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{array}{rrr} 1 & 2 & 3 \\ 0 & -2 & 2 \\ \hline 1 & 0 & 5 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Alternative approach

$$[A \mid I] \to [B \mid C]$$

$$\boxed{\overbrace{E_3 E_2 E_1}^{C} A} = B$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

#**23** Express the matrix and its inverse as products of elementary matrices.

$A = \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$ $\qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Reduce A to I_2, keeping track of E_i

$R_2 \to R_2 + \frac{2}{3}R_1$

$$\begin{array}{cc} 2 & 2 \\ -1 & 2/3 \\ \hline 0 & 8/3 \end{array}$$

$\begin{bmatrix} -3 & 1 \\ 0 & 8/3 \end{bmatrix}$ $\qquad E_1 = \begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix} \Rightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -2/3 & 1 \end{bmatrix}$

$R_1 \to R_1 - \frac{3}{8}R_2$

$$\begin{array}{cc} -3 & 1 \\ 0 & -1 \\ \hline -3 & 0 \end{array}$$

$\begin{bmatrix} -3 & 0 \\ 0 & 8/3 \end{bmatrix}$ $\qquad E_2 = \begin{bmatrix} 1 & -3/8 \\ 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 3/8 \\ 0 & 1 \end{bmatrix}$

$R_1 \to -\frac{1}{3}R_1$

$E_3 = \begin{bmatrix} -1/3 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 8/3 \end{bmatrix}$

$R_2 \to \frac{3}{8}R_2$

$E_4 = \begin{bmatrix} 1 & 0 \\ 0 & 3/8 \end{bmatrix} \qquad E_4^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 8/3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$E_4E_3E_2E_1A = I \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3/8 \end{bmatrix}\begin{bmatrix} -1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3/8 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 1 & 0 \\ -2/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3/8 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 8/3 \end{bmatrix}$$

#33 Prove that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A.

Let $E_1, E_2, \ldots, E_r$ be elementary matrices such that when the associated row operations are applied to A, the result is B.

Then $E_r \cdots E_2 E_1 A = B$. We know E_i^{-1} exist and are also elementary matrices.

$$A = E_1^{-1} E_2^{-1} \cdots E_r^{-1} B.$$

A is thus obtained from B using the row operations represented by E_i^{-1}.