1.4 – Inverses; Algebraic Properties of Matrices

Theorem 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid for matrices A, B, and C and scalars a, b, and c.

a) $A + B = B + A$ (commutative law for matrix addition)
b) $A + (B + C) = (A + B) + C = A + B + C$ (associative law for matrix addition)
c) $(AB)C = A(BC) = ABC$ (associative law for matrix multiplication)
d) $A(B + C) = AB + AC$ (left distributive law)
e) $(B + C)A = BA + CA$ (right distributive law)
f) $A(B - C) = AB - AC$
g) $(B - C)A = BA - CA$
h) $a(B + C) = aB + aC$
i) $a(B - C) = aB - aC$
j) $(a + b)C = aC + bC$
k) $(a - b)C = aC - bC$
l) $a(bC) = (ab)C$
m) $a(BC) = (aB)C = B(aC)$

Matrix multiplication is not commutative.

Consider $\underset{2\times 3}{A}\ \underset{3\times 2}{B} \to 2\times 2$

$\underset{3\times 2}{B}\ \underset{2\times 3}{A} \to 3\times 3$

$\underset{3\times 3}{A}\ \underset{3\times 2}{B}$ $\quad$ $\underset{3\times 2}{B}\ \underset{3\times 3}{A}$ X

Pf (h): For $a(B+C)$ to be equal to $aB+aC$, they must have the same size and equal entries. If $B+C$ is defined (as assumed), then they have the same size, say $m \times n$. Scalar multiplication does not change the size of a matrix, so $a(B+C)$, aB, and aC are all $m \times n$ matrices. Thus $aB+aC$ is an $m\times n$ matrix.

$(a(B+C))_{ij}$ is the entry in the i^{th} row and j^{th} column of the matrix on the left-hand side.

scalar multiplication, matrix addition

$(a(B+C))_{ij} = a(B+C)_{ij} = a(b_{ij}+c_{ij})$

$= ab_{ij} + ac_{ij}$ by the dist. prop. of real #s

$= (aB)_{ij} + (aC)_{ij}$ def of scalar mult.

Since corresponding entries are equal,

$a(B+C) = aB + aC.$

Definition: In general, $AB \neq BA$. In the special cases where $AB = BA$, we say that A and B **commute**.

Definition: A **zero matrix**, denoted O, is a matrix whose entries are all zero.

Theorem 1.4.2 Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices A and O are such that the operations can be performed, then:

a) $A + O = O + A = A$

b) $A - O = A$

c) $A - A = A + (-A) = O$

d) $OA = O$

e) If $cA = O$, then $c = 0$ or $A = O$

Disprove if $AC = BC$, then $A = B$. (Counterexample)

Let $C = O$ and let $A \neq B$.

We can have $A \neq O$, $B \neq O$ but $AB = O$.

Also, if $AC = BC$, it does not always follow that $A = B$.

Definition: **Identity matrices** are square matrices with 1's on the main diagonal and zeros everywhere else. They are denoted I or I_n if referencing the size, $n \times n$.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

Representing matrix multiplication:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{1j} & b_{1n} \\ b_{21} & b_{22} & b_{2j} & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k1} & b_{k2} & b_{kj} & b_{kn} \end{bmatrix}$$

$m \times k$ $\qquad\qquad$ $k \times n$

$$(AB)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ik} b_{kj}$$

$$= \sum_{r=1}^{k} a_{ir} b_{rj}$$ ← This is what's in the i^{th} row and j^{th} column of AB.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find $AI_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2 A$$

Don't memorize the #

Theorem 1.4.3 If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has at least one row of zeros or R is the identity matrix I_n.

pf: An $n\times n$ matrix has at most n pivots. If there are n pivots in R, then $R = I_n$. If there are fewer than n pivots, then at least one row does not have a leading 1. So at least one row is all zeros.

(uses def of pivot, def of rref)

Definition: If A is a square matrix, and if there exists a matrix B of the same size for which $AB = BA = I$, then A is said to be **invertible** (or **nonsingular**) and B is called the **inverse** of A. If no such matrix B exists, then A is said to be **singular**. Because the inverse of a matrix A is unique, we will denote it using A^{-1}.

Theorem 1.4.4 Uniqueness of a Matrix Inverse
If B and C are both inverses of the matrix A, then $B = C$.

If B and C are both inverses of A, then

$$B = BI = B(AC) = (BA)C = IC = C$$

Def of I | C is an inverse of A | Assoc prop of matrix mult. | B is an inverse of A | Def of I.

Theorem 1.4.5 The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

#8 Use Theorem 1.4.5 to compute the inverse.

$\begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$ $\quad ad - bc = 6(-1) - (-2)(4) = 2$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & -4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -1/2 & -2 \\ 1 & 3 \end{bmatrix}$$

Check: $A^{-1}A = \begin{bmatrix} -1/2 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Theorem 1.4.6 If $\boxed{A \text{ and } B \text{ are invertible}}$ matrices with the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

A^{-1}, B^{-1} exist.

pf: Consider $\boxed{(AB)}\boxed{(B^{-1}A^{-1})}$

$= A(BB^{-1})A^{-1}$ assoc. prop.

$= AIA^{-1}$ Def of inverse

$= AA^{-1}$ Def of I

$= I$ Def of inverse

Since multiplying (AB) by $(B^{-1}A^{-1})$ yields I,

$(AB)^{-1} = B^{-1}A^{-1}$

Theorem 1.4.7 Properties of Negative Exponents

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$.

b) A^n is invertible and $(A^n)^{-1} = A^{-n} = \left(A^{-1}\right)^n$.

c) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.

reciprocal

inverse

#15 Use the given information to find A.

$$(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$$

(c) $\frac{1}{7}A^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} -21 & 49 \\ 7 & -14 \end{bmatrix}$

(a) $A = \begin{bmatrix} -21 & 49 \\ 7 & -14 \end{bmatrix}^{-1}$

$$A = \frac{1}{21(14) - 7(49)}\begin{bmatrix} -14 & -49 \\ -7 & -21 \end{bmatrix} = \begin{bmatrix} 2/7 & 1 \\ 1/7 & 3/7 \end{bmatrix}$$

#23 Find all values of a, b, c, and d (if any) for which the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ commute. $AB = BA$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

$$AB = BA \Rightarrow \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \Rightarrow c = 0,\ a = d,\ b \in \mathbb{R}.$$

#25 Solve the system using an inverse matrix.

$3x_1 - 2x_2 = -1$

$4x_1 + 5x_2 = 3$

Matrix equation (form $A\vec{x} = \vec{b}$)

$A^{-1}A\vec{x} = A^{-1}\vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$

$$\begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{15+8}\begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix}$$

$$\vec{x} = \frac{1}{23}\begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\vec{x} = \frac{1}{23}\begin{bmatrix} 1 \\ 13 \end{bmatrix}$$

$\frac{1}{3} 3x = 15 \frac{1}{3}$

$x = 5$

$$(x_1, x_2) = \left(\frac{1}{23}, \frac{13}{23}\right)$$

#39 Simplify the expression assuming that A, B, C and D are invertible.
$(AB)^{-1}\left(AC^{-1}\right)\left(D^{-1}C^{-1}\right)^{-1}D^{-1}$

$B^{-1}(A^{-1}A)(C^{-1}C)(D\,D^{-1}) = B^{-1}$

Theorem 1.4.8 Properties of the Transpose
If the sizes of the matrices are such that the stated operations can be performed, then:

a) $\left(A^T\right)^T = A$
b) $(A+B)^T = A^T + B^T$
c) $(A-B)^T = A^T - B^T$
d) $(kA)^T = kA^T$
e) $(AB)^T = B^T A^T$

$m\times k \quad k\times n \quad n\times k \quad k\times m$

A is a matrix
$[a_{ij}]$ is a matrix
$(A)_{ij}$ is an entry (position)
a_{ij} is an entry (real #)

pf (b) Let A and B be matrices of the same size. $((A+B)^T)_{ij} = (A+B)_{ji} = (A)_{ji} + (B)_{ji}$

$= (A^T)_{ij} + (B^T)_{ij}$

$= (A^T + B^T)_{ij}$

You try

So $(A+B)^T = A^T + B^T$

Theorem 1.4.9 If A is an invertible matrix, then A^T is also invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.