1.2 – Gaussian Elimination

#7 Solve the system by Gaussian elimination.

$$x - y + 2z - w = -1$$
$$2x + y - 2z - 2w = -2$$
$$-x + 2y - 4z + w = 1$$
$$3x \qquad\qquad - 3w = -3$$

→ using an augmented matrix

0 zero

∅ empty set : $\emptyset = \{\ \}$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

1 2

$R_2 \to R_2 - 2R_1$

2	1	-2	-2	-2
-2	2	-4	2	2
0	3	-6	0	0

$R_3 \to R_3 + R_1$

-1	2	-4	1	1
1	-1	2	-1	-1
0	1	-2	0	0

$R_4 \to R_4 - 3R_1$

3	0	0	-3	-3
-3	3	-6	3	3
0	3	-6	0	0

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$R_4 \to R_4 - 3R_2$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is now

① $x - y + 2z - w = -1$

② $y - 2z = 0$

① $x = y - 2z + w - 1$

② $y = 2z$

Substituting yields

$x = w - 1$

$y = 2z$

This is in row echelon form via Gaussian elimination

solution: $x = w - 1$, $y = 2z$

$(x, y, z, w) = (w-1, 2z, z, w)$

Let $s = z$, $t = w$. Then

$x = t-1$
$y = 2s$
$z = s$
$w = t$

$(x, y, z, w) = (t-1, 2s, s, t)$

Definitions: Row echelon form (#1-3 below) and **reduced row echelon form** (#1-4 below) of a matrix – to be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. This is a **leading 1.**
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

Back to $\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$R_1 \to R_1 + R_2$

$$\begin{array}{ccccc} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 & -1 \end{array}$$

$$\begin{array}{ccccc} x & y & z & w & \text{const} \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x - w = -1 \implies x = w - 1 \\ \to y - 2z = 0 \qquad\ \ y = 2z \end{array}$$

Let $s = z$, $t = w$.

same result as before.

Reduced row echelon form
via Gauss-Jordan elimination

Leading 1s are in the x- and y-columns,
So x & y are leading variables.
The remaining variables are free variables.

Definitions: **Leading variables** correspond to the leading 1's in the augmented matrix for a linear system. **Free variables** are the remaining variables in the linear system.

Every variable in a system is either leading or free.

Definition: In a **homogeneous linear system**, the constant terms are all zero. That is, it has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

Theorem 1.2.1 Free Variable Theorem for Homogeneous Systems
If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

$2x+3y-5z=0$
$4x+8y-7z=0$

correspond to leading 1s.

Theorem 1.2.2 A homogeneous linear system with more unknowns than equations has infinitely many solutions.

pf: The # of equations in a system is the # of rows in its augmented matrix. The # of leading 1s is no more than the number of rows. Thus, such a system has at least 1 free variable and so has infinitely many solutions.

#28 What condition, if any, must a, b, and c satisfy for the linear system to be consistent?

$$x + 3y + z = a$$
$$-x - 2y + z = b$$
$$3x + 7y - z = c$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{array}\right]$$

1 2

$R_2 \to R_2 + R_1$

$$\begin{array}{cccc} -1 & -2 & 1 & b \\ 1 & 3 & 1 & a \\ \hline 0 & 1 & 2 & a+b \end{array}$$

$R_3 \to R_3 - 3R_1$

$$\begin{array}{cccc} 3 & 7 & -1 & c \\ -3 & -9 & -3 & -3a \\ \hline 0 & -2 & -4 & -3a+c \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & -3a+c \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\begin{array}{cccc} 0 & -2 & -4 & -3a+c \\ 0 & 2 & 4 & 2a+2b \\ \hline 0 & 0 & 0 & -a+2b+c \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & -a+2b+c \end{array}\right] \to R_3 : 0 = -a+2b+c$$

the system is consistent if $a = 2b + c$.

Solve the given linear system by any method.

#16

$$2x - y - 3z = 0$$
$$-x + 2y - 3z = 0$$
$$x + y + 4z = 0$$

$$\left[\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]$$

writing the zeros is optional

1 2

$R_2 \to R_1 + 2R_2$

$$\begin{array}{ccc} 2 & -1 & -3 \\ -2 & 4 & -6 \\ \hline 0 & 3 & -9 \end{array}$$

$R_2 \to \frac{1}{3}R_2$

$R_3 \to R_3 + R_2$

$$\begin{array}{ccc} 1 & 1 & 4 \\ -1 & 2 & -3 \\ \hline 0 & 3 & 1 \end{array}$$

$$\begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & -3 \\ 0 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{array}{ccc} 0 & 3 & 1 \\ 0 & -3 & 9 \\ \hline 0 & 0 & 10 \end{array}$$

$\to R_3 \to \frac{1}{10}R_3$

4 3

$$\begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 3R_3$

$$\begin{array}{ccc} 0 & 1 & -3 \\ 0 & 0 & 3 \\ \hline 0 & 1 & 0 \end{array}$$

$R_1 \to R_1 + 3R_3$

$$\begin{array}{ccc} 2 & -1 & -3 \\ 0 & 0 & 3 \\ \hline 2 & -1 & 0 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{array}{ccc} 2 & -1 & 0 \\ 0 & 1 & 0 \\ \hline 2 & 0 & 0 \end{array}$$

$R_1 \to \frac{1}{2}R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$x = 0$
$y = 0$
$z = 0$

$(x, y, z) = (0, 0, 0)$

↑ this is the <u>trivial solution</u>.

#19

$$2x + 2y + 4z = 0 \Rightarrow \text{use } x + y + 2z = 0$$
$$w - y - 3z = 0$$
$$2w + 3x + y + z = 0$$
$$-2w + x + 3y - 2z = 0$$

$$\begin{array}{c} \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix} \\ \begin{array}{cccc} 1 & 2 & 3 & \end{array} \end{array} \quad R_1 \Leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix} \quad R_3 \to R_3 - 2R_1 \quad R_4 \to R_4 + R_3$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 7 \\ 0 & 4 & 4 & -1 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 - 4R_2 \end{array} \quad \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row echelon form

$$\begin{array}{cccc|c} w & x & y & z & \\ \hline 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

Reduced row echelon form (rref)

$$w - y = 0$$
$$x + y = 0$$
$$z = 0$$

w, x, z are leading variables. y is a free variable

Let $t = y$. Then $w = t$, $x = -t$, $z = 0$

solution: $(t, -t, t, 0)$

#23 The augmented matrix for a linear system is given in which the asterisk represents an unspecified real number. Determine whether the system is consistent, and if so whether the solution is unique. Answer "inconclusive" if there is not enough information to make a decision.

a. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$ Consistent, unique solution

b. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$ Consistent
Not unique (free variable)

c. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\rightarrow$ contradiction $\Rightarrow$ no solution

d. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{array}\right]$ $\rightarrow$ $z = 0$

Inconclusive.

If $* = 0$, consistent.

If $* \neq 0$, inconsistent.